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Generalized even and odd coherent states of a single bosonic mode

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Abstract. Some properties of the eigensolutions of two commuting non Hermitian operators are brought to the light and exploited to introduce a novel characterization of a set of non classical states of a single bosonic mode. In this context, the construction of possible generalizations of the even and odd coherent states is developed in detail.

PACS. 42.50.-p Quantum optics – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

In this paper we present the construction of a class of normalizable non classical states relative to a single bosonic mode characterized by an energy raising operator $a^\dagger$ and an energy lowering operator a obeying the commutation relation $[a, a^\dagger] = 1$. The one-dimensional quantum harmonic oscillator of frequency Ω is described by the Hamiltonian

$$H = \Omega\left(a^\dagger a + \frac{1}{2}\right) \qquad (1)$$

and, undoubtedly, represents the simplest and most extensively studied system whose dynamics may be related to the time evolution of the non Hermitian dimensionless bosonic variables a and $a^\dagger$.

Several other systems may indeed be bosonized. This means that, under appropriate conditions depending on the physical situation under scrutiny, the physical behaviour of these systems may be investigated adopting the Hamiltonian model (1) together with an effective definition of Ω.

The quantum mechanical motion of the center of mass of a trapped ion [1,2], the quantized electro magnetic field of a single-mode cavity [3], the low-temperature quantum dynamics of a fictitious phase-particle at the bottom of the Josephson potential [4] and, in general, material systems making small oscillations about a stable equilibrium point, provide concrete examples of fruitfully bosonizable physical systems.

Quite recently the orthonormalized eigenstates of an arbitrarily prefixed power of the annihilation operator a have been introduced [5]. Some interesting non classical properties of these states such as, for instance, anti bunching and higher order squeezing have been studied [5,6]. It should be however noted that both these references confine themselves to define these eigenstates of a^k $(k \geq 3)$ intuitively generalizing the well-known Fock representation of the even and odd coherent states introduced by Dodonov *et al.* [7].

Surprisingly enough, moreover, the authors do no attempt to establish eventual physical links between the states they present and the even and odd coherent states. The scope of this paper is twofold. Firstly we wish to provide a novel constructive way for characterizing this set of non classical states of a single bosonic mode (hereafter also referred to as single-mode cavity field). In particular we deduce their explicit Fock and coherent expansions defining and solving an appropriate eigenvalue problem. Secondly, in the context of this approach, we look for and bring to the light the existence of a physical property which paves the way for defining and recognizing possible generalizations of the so-called even and odd coherent states in the set of the eigenstate of a^k whatever k is.

It is well-known that if $|\alpha\rangle$ $(\alpha \in C, |\alpha| \neq 0)$ is any coherent state of a quantized single-mode cavity field, the two orthogonal states

$$|\alpha; 2, j\rangle = N^{(2)}_{\alpha,j}(|\alpha\rangle + e^{i\pi j}|-\alpha\rangle) \quad j = 0,\ 1 \qquad (2)$$

are normalized eigenstates of a^2,

$$N^{(2)}_{\alpha,j} = \frac{1}{\sqrt{2}}\left[1 + (-1)^j e^{-2|\alpha|^2}\right]^{-\frac{1}{2}} \qquad (3)$$

being the appropriate normalization constants. The state $|\alpha; 2, 0\rangle$ $(|\alpha; 2, 1\rangle)$ is called even (odd) coherent state because $\langle 2s+1|\alpha; 2, 0\rangle = 0$ $(\langle 2s|\alpha; 2, 1\rangle = 0)$ for any Fock state $|2s+1\rangle$ $(|2s\rangle)$ of the cavity field, described by equation (1), containing an odd (even) number of excitations.

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The two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, with $\alpha \neq 0$, are degenerate eigenstates of a^2 pertaining to the same eigenvalue α^2. Thus, all the linear combination of two 180°-out of phase coherent states are eigenstates of a^2. In particular, the well-studied non classical [8,9] symmetric and antisymmetric superpositions of $|\alpha\rangle$ and $|-\alpha\rangle$, prototype of the so-called Schrödinger cat states [10], belong to the same invariant subspace of a^2.

Consider now the non Hermitian operator $a^2 e^{i\pi a^\dagger a}$. It commutes with a^2 and, in the invariant subspace of a^2 correspondent to $\alpha^2 \neq 0$, its eigenstates are $|\alpha; 2, 0\rangle$ and $|\alpha; 2, 1\rangle$ whereas the relative eigenvalues are $\gamma_0^{(2)} = \alpha^2$ and $\gamma_1^{(2)} = -\alpha^2$. Therefore, looking for the eigensolutions of the non Hermitian operator $a^2 e^{i\pi a^\dagger a}$ yields a natural way of building up linear combinations of coherent states characterized by the fact that the difference between the numbers of quanta present in two successive Fock states of their respective number representations is fixed and equal to 2. (Sometimes we refer to this difference calling it "distance").

The occurrence of such peculiar oscillations may be interpreted as a quantum interference effect in the phase space and therefore turns out to be a non classical signature of these states [11,12]. On the basis of these considerations, it might be of interest to find a general way for characterizing states of the single-mode quantized cavity field whose Fock representations exhibit a fixed and equal to $n \geq 2$ distance between two successive energy eigenvectors.

To this end we introduce the non Hermitian operators

$$C_n = a^n e^{i\frac{2\pi}{n} a^\dagger a}, \; n = 1, \; 2, \; \ldots \tag{4}$$

The eigenstates of C_1 are the coherent states and the eigenstates of C_2, relative to generic not null eigenvalues, are the even or the odd coherent states.

The eigenstates of a^n are all the coherent states $|\alpha\rangle$ as well as the Fock states $|1\rangle, |2\rangle, \ldots, |n-1\rangle$. Since $a^n|\alpha\rangle = \alpha^n|\alpha\rangle$ and $a^n|s\rangle = 0$ when $s = 0, \; 1, \ldots, \; n-1$, each eigenvalue of a^n is n-fold degenerate. This fact, in analogy with the properties of the eigensolutions of C_2, suggests seeking orthonormal basis of eigenstates of C_n in each invariant subspace of a^n.

If $\alpha = 0$ the solution of this problem is immediate because the n Fock states $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ are eigenstates of a^n and C_n pertaining to their common eigenvalue 0.

For any fixed $\alpha \neq 0$ let's represent symbolically the n simultaneous eigenstates of a^n and C_n by $|\alpha; n, j\rangle$ where $j = 0, 1, \ldots, n-1$ is a label introduced at this stage for convenience of classification. Thus, by definition, we put:

$$a^n|\alpha; n, j\rangle = \alpha^n|\alpha; n, j\rangle \tag{5}$$

$$C_n|\alpha; n, j\rangle = \gamma_j^{(n)}|\alpha; n, j\rangle \tag{6}$$

$$\langle\alpha; n, j|\alpha; n, j'\rangle = \delta_{jj'} \tag{7}$$

where $\gamma_j^{(n)}$ denotes a complex unknown eigenvalue of C_n.

We underline that the states solutions of this problem provide particular examples of the multi photon coherent states defined by Jex and Buzek [13].

Applying the unitary operator $e^{i\frac{2\pi}{n}a^\dagger a}$($e^{-i\frac{2\pi}{n}a^\dagger a}$) to both members of equation (5) (Eq. (6)), we deduce that the states $|\alpha; n, j\rangle$ must also be solutions of the equation

$$e^{i\frac{2\pi}{n}a^\dagger a}|\alpha; n, j\rangle = \left(\frac{\gamma_j^{(n)}}{\alpha^n}\right)|\alpha; n, j\rangle. \tag{8}$$

Thus we see that for any fixed $\alpha \neq 0$, the simultaneous eigenstates of a^n and C_n are necessarily eigensolutions of the unitary operator $e^{i\frac{2\pi}{n}a^\dagger a}$. The eigenvalues of this operator are the nth roots of 1 and may be represented as $\varepsilon_j^{(n)} = \exp\left(i\frac{2\pi}{n}j\right)$ where $j = 0, 1, \ldots, n-1$. Thus we find that $\gamma_j^{(n)}$ and α^n are related as follows:

$$\gamma_j^{(n)} = \alpha^n e^{i\frac{2\pi}{n}j}. \tag{9}$$

The eigenvectors of $e^{i\frac{2\pi}{n}a^\dagger a}$ pertaining to $\varepsilon_j^{(n)}$ are the infinitely-many Fock states $|hn + j\rangle$ where $h = 0, 1, \ldots$ is an arbitrary natural number. This means that the number state representation of $|\alpha; n, j\rangle$ may be from the beginning conveniently written as:

$$|\alpha; n, j\rangle = \tilde{N}_{\alpha,j}^{(n)} \sum_{h=0}^{\infty} b_{\alpha,j}^{(n)}(h)|hn + j\rangle \tag{10}$$

where $j = 0, 1, \ldots, n-1$, and $\tilde{N}_{\alpha,j}^{(n)} > 0$ is an appropriate normalization constant. In view of the fundamental theorem on the division between integers [14], the diophantine equation $hn + j = h'n + j'$ in the natural unknowns h and h', is impossible under the condition $j, j' = 0, 1, \ldots, n-1$ and $j \neq j'$. This amounts to saying that searching for solutions of equations (5, 6) in the form expressed by equation (10) satisfies automatically the orthonormality condition (7).

It is worth remarking here that the Fock state representation (10) of $|\alpha; n, j\rangle$, consists in an arithmetic infinite progression having the state $|j\rangle$ as initial term and a common difference, equal to n, between successive terms.

To find the h-dependence of $b_{\alpha,j}^{(n)}(h)$, we insert equation (10) into equation (6) and exploit equation (9) getting the following linear difference equation of order 1 with variable coefficients:

$$b_{\alpha,j}^{(n)}(h+1) = \alpha^n \sqrt{\frac{(hn+j)!}{((h+1)n+j)!}}\, b_{\alpha,j}^{(n)}(h) \tag{11}$$

which must be associated to an appropriately chosen initial condition.

In view of equations (7, 11), $b_{\alpha,j}^{(n)}(0)$ cannot vanish. On the other hand, considering that equation (10) contains an adjustable normalization constant, it is of no relevance the particular choice of the not null value attributed to

$$|\alpha;n,j\rangle = \tilde{N}_{\alpha,j}^{(n)} \sum_{h=0}^{\infty} \frac{\alpha^{nh+j}}{\sqrt{(hn+j)!}} |hn+j\rangle = \tilde{N}_{\alpha,j}^{(n)} \sum_{s=0}^{\infty} \frac{\alpha^s}{\sqrt{s!}} \delta_{[\frac{s-j}{n}],\frac{s-j}{n}} |s\rangle$$
$$= \frac{\tilde{N}_{\alpha,j}^{(n)}}{n} \sum_{s=0}^{\infty} \frac{\alpha^s}{\sqrt{s!}} \left(\sum_{r=0}^{n-1} (\varepsilon_r^{(n)})^{s-j} \right) |s\rangle = \frac{\tilde{N}_{\alpha,j}^{(n)}}{n} e^{\frac{|\alpha|^2}{2}} \sum_{r=0}^{n-1} \left(\varepsilon_r^{(n)}\right)^{-j} |\alpha\varepsilon_r^{(n)}\rangle \tag{18}$$

$b_{\alpha,j}^{(n)}(0)$. Thus we put $b_{\alpha,j}^{(n)}(0) = \alpha^j/\sqrt{j!}$. It is not difficult to prove by mathematical induction that then

$$b_{\alpha,j}^{(n)}(h) = \frac{\alpha^{nh+j}}{\sqrt{(hn+j)!}} \cdot \tag{12}$$

Since the series

$$\sum_{h=0}^{\infty} |b_{\alpha,j}^{(n)}(h)|^2 = \sum_{h=0}^{\infty} \frac{|\alpha|^{2(nh+j)}}{(hn+j)!} \tag{13}$$

is convergent whatever $\alpha \in C$ and $j = 0, 1, \ldots, n-1$ are, then the n states

$$|\alpha;n,j\rangle = \tilde{N}_{\alpha,j}^{(n)} \sum_{h=0}^{\infty} \frac{\alpha^{nh+j}}{\sqrt{(hn+j)!}} |hn+j\rangle \qquad j = 0,\ 1,\ n-1 \tag{14}$$

solve the problem posed by equations (5–7) provided that

$$\tilde{N}_{\alpha,j}^{(n)} = \left(\sum_{h=0}^{\infty} |b_{\alpha,j}^{(n)}(h)|^2 \right)^{-\frac{1}{2}}. \tag{15}$$

Having in mind to bring to the light the eventual link between the even and odd coherent states of the single bosonic mode system and the states $|\alpha;n,j\rangle$, we look for the coherent state expansion of these vectors in the invariant subspace of a^n relative to its eigenvalue α^n.

To this end, consider the n normalized eigenstates of $a^n \left\{ |\alpha\varepsilon_j^{(n)}\rangle, a^n|\alpha\varepsilon_j^{(n)}\rangle = \alpha^n |\alpha\varepsilon_j^{(n)}\rangle, j = 0,1,\ldots,n-1 \right\}$ and the Fock expansion of the coherent state $|\alpha\varepsilon_j^{(n)}\rangle$

$$|\alpha\varepsilon_j^{(n)}\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{s=0}^{\infty} \frac{(\alpha\varepsilon_j^{(n)})^s}{\sqrt{s!}} |s\rangle \cdot \tag{16}$$

With the help of the well-known identity, ($[x]$=integer part of x),

$$\sum_{r=0}^{n-1} (\varepsilon_r^{(n)})^p = n\delta_{\left[\frac{p}{n}\right],\frac{p}{n}} \qquad p = 0, \pm1, \pm2, \ldots \tag{17}$$

we succeed in transforming the expression (14) for $|\alpha;n,j\rangle$ ($\alpha \neq 0$) as follows:

see equation (18) above

where equation (16) has been used. Apart from its inherent theoretical interest on which we shall comment later on, this result proves itself to be useful from a mathematical point of view too. We note, in fact, that equation (18), expressing the normalized state $|\alpha;n,j\rangle$ as linear combination of n coherent states only, provides a successfully starting point for attaining an exact explicit expression of $\tilde{N}_{\alpha,j}^{(n)}$. From equations (7, 18) and with the help of the formula

$$\langle \alpha\epsilon_{r'}^{(n)} | \alpha\epsilon_r^{(n)} \rangle = e^{-2|\alpha|^2 \sin^2\left(\frac{\pi}{n}(r-r')\right)} \times e^{i|\alpha|^2 \sin\left(\frac{2\pi}{n}(r-r')\right)} \tag{19}$$

we obtain

$$\tilde{N}_{\alpha,j}^{(n)} = n e^{-\frac{1}{2}|\alpha|^2} \Big| \sum_{r,r'=0}^{n-1} e^{-i\frac{2\pi}{n}(r-r')j} \times e^{-2|\alpha|^2 \sin^2\left(\frac{\pi}{n}(r,r')\right)} e^{i|\alpha|^2 \sin\left(\frac{2\pi}{n}(r,r')\right)} \Big|^{-\frac{1}{2}}. \tag{20}$$

To take advantage from the fact that the argument in the finite double sum appearing in equation (20) is a function of $(r-r')$ only, we have built up the following reduction formula of general validity:

$$\sum_{r,r'=0}^{n-1} f(r,r') = \sum_{r_-=-(n-1)}^{n-1} \sum_{r_+=|r_-|}^{2(n-1)-|r_-|} F(r_-,r_+) \times \frac{\left(1+(-1)^{r_++|r_-|}\right)}{2} = \sum_{r_-=-(n-1)}^{n-1} \sum_{\tilde{r}_+=0}^{(n-1)-|r_-|} G(r_-,\tilde{r}_+) \tag{21}$$

where

$$\begin{cases} r_- &= r - r' \\ r_+ &= r + r' = 2\tilde{r}_+ + |r_-| \\ f(r,r') &= f\left(\frac{r_++r_-}{2}, \frac{r_+-r_-}{2}\right) \equiv F(r_-,r_+) \\ &= F(r_-, 2\tilde{r}_+ + |r_-|) \equiv G(r_-,\tilde{r}_+) \end{cases}. \tag{22}$$

$$|\alpha;n,j\rangle = \sqrt{n}e^{-|\alpha|^2/2}\left[1+(-1)^j e^{-2|\alpha|^2}\delta_{2[\frac{n}{2}],n} + \tilde{\delta}_{\bar{n},0}\sum_{\nu=1}^{\bar{n}} e^{-2|\alpha|^2\sin^2\frac{\pi}{n}\nu}\cos\left[|\alpha|^2\sin\frac{2\pi}{n}\nu - \frac{2\pi}{n}j\nu\right]\right]^{-\frac{1}{2}}\sum_{h=0}^{\infty}\frac{\alpha^{nh+j}}{\sqrt{(nh+j)!}}|nh+j\rangle \quad (25)$$

Exploiting this mathematical tool yields the following expression for $\tilde{N}^{(n)}_{\alpha,j}$:

$$\tilde{N}^{(n)}_{\alpha,j} = \sqrt{n}e^{-|\alpha|^2/2}\left[1+(-1)^j e^{-2|\alpha|^2}\delta_{2[\frac{n}{2}],n} + \tilde{\delta}_{\bar{n},0}\sum_{\nu=1}^{\bar{n}} e^{-2|\alpha|^2\sin^2\frac{\pi}{n}\nu} \times\cos\left[|\alpha|^2\sin\frac{2\pi}{n}\nu - \frac{2\pi}{n}j\nu\right]\right]^{-\frac{1}{2}} \quad (23)$$

where $2\bar{n} = n-1-\delta_{2[\frac{n}{2}],n}$ and $\tilde{\delta}_{\bar{n},0} = 2(1-\delta_{\bar{n},0})$.

For the sake of comparison, it is useful to write down the relation between $\tilde{N}^{(n)}_{\alpha,j}$ and the normalization constant $N^{(n)}_{\alpha,j}$ relative to the equal-weighted coherent state superposition appearing in the last member of equation (18):

$$N^{(n)}_{\alpha,j} = \frac{e^{\frac{1}{2}|\alpha|^2}}{n}\tilde{N}^{(n)}_{\alpha,j}. \quad (24)$$

We note that for $n=2$, we recover from equations (23, 24) the correct normalization constants, given by equation (3), relative to the even ($j=0$) and odd ($j=1$) coherent states.

Inserting equation (23) into equation (14), completes the construction of the simultaneous eigenstates of the non Hermitian operators a^n and C_n yielding

see equation (25) above.

Equation (25) shows that each state $|\alpha;n,j\rangle$ may be formally obtained by extracting an arithmetic progression of terms from the Fock expansion of the coherent state $|\alpha\rangle$. At the same time equation (18) says that such non classical "extracted states" may be expressed as linear combinations of n equal-weighted coherent states positioned at equal distance on a circle of radius $|\alpha|$ in the phase space. For these reasons we propose to call the states defined by equation (25) *subcoherent states of order n.*

It is worth noting that when the coherent state superpositions defined by equation (18), are taken on circles with an high enough n-dependent radius, then these subcoherent states $|\alpha;n,j\rangle$ describe generalized Schrödinger cat states in the sense that they may be regarded as linear combinations of n macroscopically distinguishable states.

A characteristic feature of all the states $|\alpha;n,j\rangle$, once more easily deducible from equation (18), is that the phase difference $\delta^{(n)}_j$ between two successive amplitudes $(\varepsilon^{(n)}_r)^{-j}$ and $(\varepsilon^{(n)}_{r+1})^{-j}$ is constant and equal to $(2\pi/n)j$. This circumstance gives us the opportunity of classifying the n different simultaneous eigenstates of a^n and C_n in terms of the n different values of $\delta^{(n)}_j$, namely 0, $(2\pi/n), \ldots, (2\pi/n)(n-1)$.

In particular, for any n, $\delta^{(n)}_0 = 0$ so that the state

$$|\alpha;n,0\rangle = \frac{e^{\frac{1}{\sqrt{2}}|\alpha|^2}}{n}\tilde{N}^{(n)}_{\alpha,0}\sum_{r=0}^{n-1}|\alpha\epsilon^{(n)}_r\rangle = \tilde{N}^{(n)}_{\alpha,0}\sum_{h=0}^{\infty}\frac{\alpha^{nh}}{\sqrt{(hn)!}}|hn\rangle \quad (26)$$

with $\tilde{N}^{(n)}_{\alpha,0}$ given by equation (23), generalizes the even coherent state $|\alpha;2,0\rangle$. Our suggestion is to call it *even coherent state of order n.*

In view of equation (2), possible generalized odd coherent states might be defined on the basis of the condition $\delta^{(n)}_j = \pi$. It is immediate to realize that such a condition may be satisfied for any even n choosing $j=n/2$, whereas it cannot be verified for any odd n. This means that, in accordance with our definition, the state $|\alpha;n,n/2\rangle$, meaningful under the prescription n even, provides a reasonable generalization of the odd coherent state $|\alpha;2,1\rangle$ and may be called *odd coherent state of order n.*

It should be noted that, if n is divisible by 4, $\langle 2s+1|\alpha;n,n/2\rangle=0$ for any s and that, if n is odd, $\sum_{s=0}^{\infty}|\langle 2s|\alpha;n,0\rangle|^2 < 1$. These properties clearly evidence that, differently from the case $n=2$, the parity of the number states appearing in the Fock expansion of $|\alpha;n,0\rangle$ or $|\alpha;n,n/2\rangle$ does not play any characterizing role when $n>2$.

Indeed our way of introducing the even or odd coherent states of order n relies on the existence of two n-independent possible values of $\delta^{(n)}_j$, namely 0 and π (n even).

In other words the sets $\{|\alpha;n,0\rangle\}$ and $\{|\alpha;n/2,n\rangle\}$, are identified by peculiar n-independent prescriptions on the way of superposing the coherent states of the class $\{|\alpha\varepsilon^{(n)}_r\rangle\}$.

Concerning the case n odd, we propose to classify as generalized odd coherent states of order n those states for which $\delta^{(n)}_j$ is near π as much as possible, that is the states $|\alpha;n,j\rangle$ correspondent to $j=(n\pm 1)/2$. We note that, in this case too, the Fock expansion of these states generally contains contributions with both even and odd number of quanta.

In order to consolidate the physical link between $|\alpha;2,0\rangle$ and $|\alpha;n,0\rangle$ on the one hand, and between $|\alpha;2,1\rangle$ and the states $|\alpha;n,n/2\rangle$ (n even) or $|\alpha;n,(n\pm 1)/2\rangle$ (n odd) on the other hand, it is of relevance to bring to the light, for each kind of generalized states, even or odd, the existence of qualitatively n-independent physical effects whose origin may be traced back to the quantum interference in the phase space.

To this end we have evaluated the influence of the interference process on the mean population of the quantized single-mode cavity field, getting

$$\langle \alpha; n, j | a^\dagger a | \alpha; n, j \rangle = |\alpha|^2 \frac{(\tilde{N}_{\alpha,j}^{(n)})^2}{(\tilde{N}_{\alpha,j-1}^{(n)})^2} \equiv P_\alpha^{(n)}(\delta_j^{(n)}) \quad (27)$$

where $\tilde{N}_{\alpha,j-1}^{(n)}$ is given by equation (23) even for $j = 0$. Writing $P_\alpha^{(n)}(\delta_j^{(n)})$ instead of $P_\alpha^{(n)}(j)$, we wish to emphasize the fact that the expectation value of $a^\dagger a$ on the state $|\alpha; n, j\rangle$ depends of j only through $(2\pi/n)j \equiv \delta_j^{(n)}$. A careful examination of the behaviour of $P_\alpha^{(n)}(\delta_j^{(n)})$ based on equations (23, 27), reveals that it exhibits a n- and $|\alpha|^2$-sensitive oscillatory dependence on the phase-difference parameter, reaching its absolute minimum for $\delta_j^{(n)} = 0$ and its absolute maximum for $\delta_j^{(n)} = \pi$ (or near π as much as possible) whatever n and α are.

In this paper we have presented a detailed new characterization of all the sub coherent states of order n relative to a single bosonic mode. We have carefully discussed some properties of these states showing the possibility of singling out in this class two particular sets of states which represent a convincing generalization of the traditional even and odd coherent states.

It is well-known that the even and odd coherent states have been realized in laboratory [15]. We wish moreover to remark that a proposal for the experimental production of the class of states introduced in this paper has been proposed quite recently in the literature [16]. We believe that the states $|\alpha; n, j\rangle$ constructed in this paper might also be generated in laboratory using some appropriate nonlinear matter-radiation coupling [9,12,17].

We wish to conclude emphasizing that, in our opinion, the novel point of view adopted in this paper to build up the states $|\alpha; n, j\rangle$, might be of some help to find and study other examples of non classical superpositions of a finite number of coherent states.

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